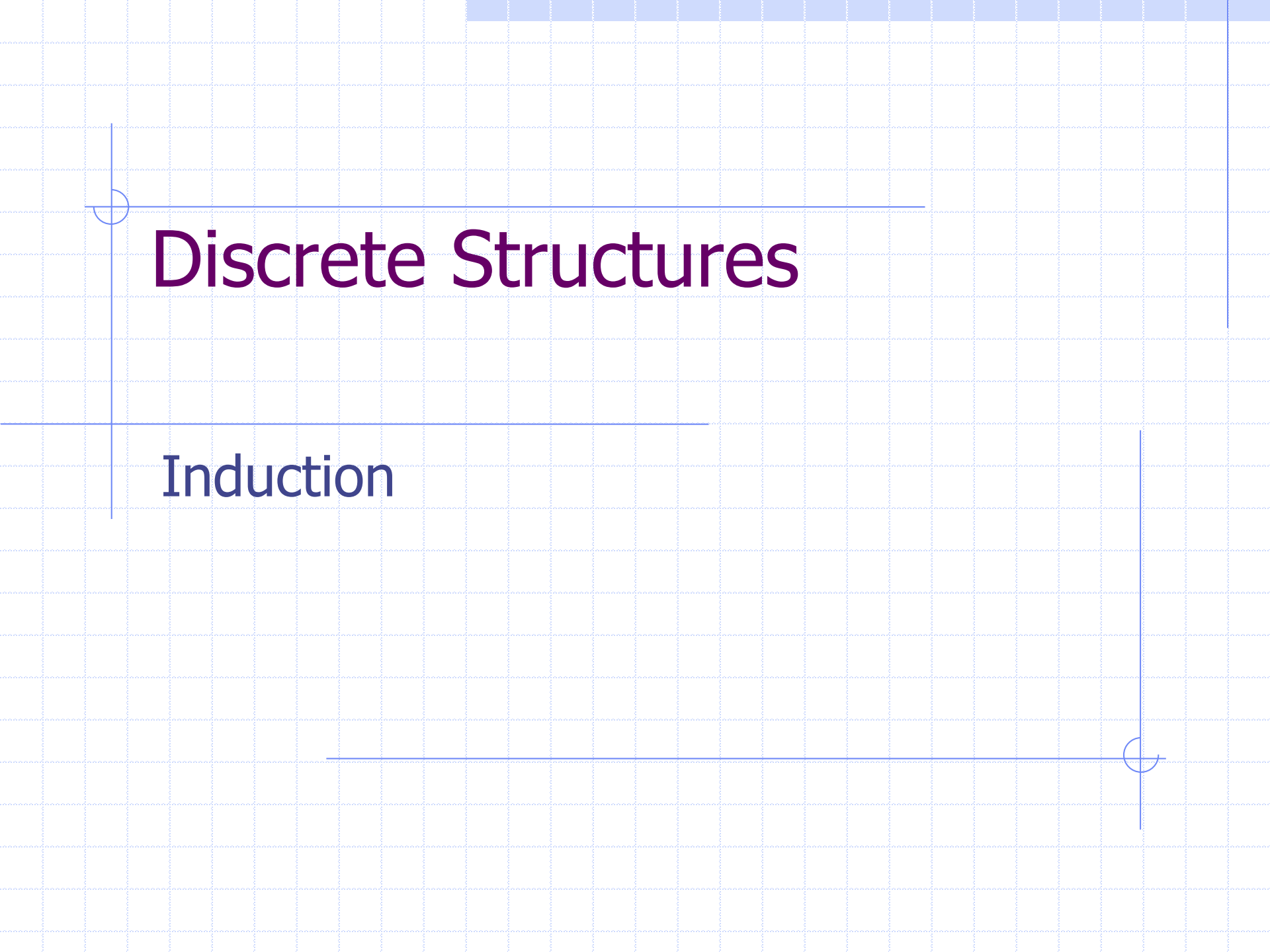
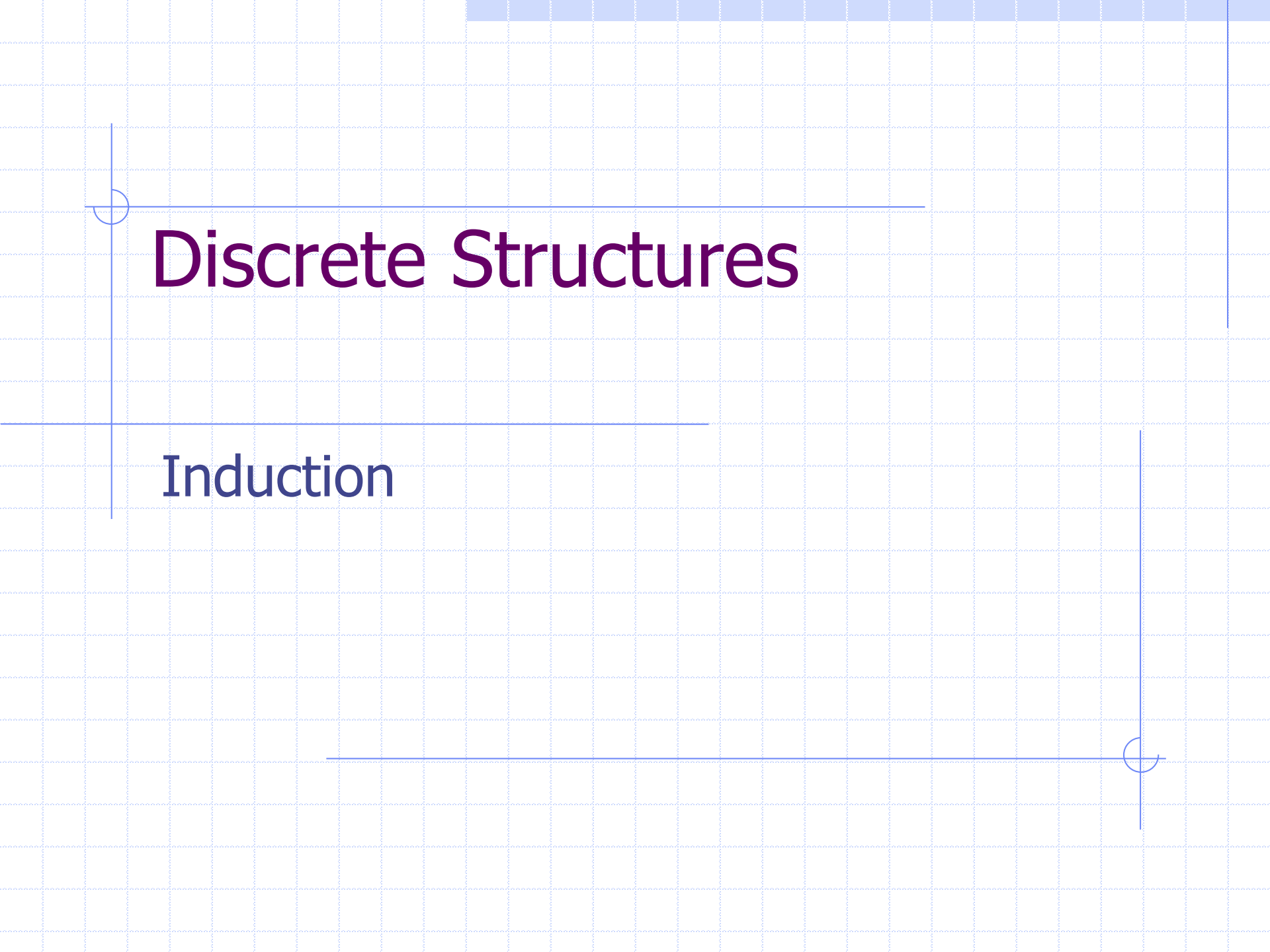




Discrete Structures

Induction



Example

- $1 = (1 \times 2)/2$
 - $1 + 2 = (2 \times 3)/2$
 - $1 + 2 + 3 = (3 \times 4)/2$
 - $1 + 2 + 3 + 4 = (4 \times 5)/2$
 - $1 + 2 + 3 + 4 + 5 = 4 \times 5/2 + 5 = 5 \times 6/2$
 - $P(n): 1 + 2 + \dots + n = n(n + 1)/2$
 - Prove $P(n + 1)$ under the assumption $P(n)$ is true
 - Indeed we prove $\forall n \geq 1 (P(n) \Rightarrow P(n + 1))$
 - Or $\forall n (P(n) \rightarrow P(n + 1))$ is tautology
-
- $1 + 2 + \dots + n + (n + 1) = n \times (n + 1)/2 + (n + 1) = (n + 1)(n + 2)/2$

Induction Principle

- Basis Step: $P(1)$ is true
- Inductive Step: $\forall n \geq 1: P(n) \Rightarrow P(n + 1)$

Assumption " $P(n)$ is true" is called **inductive hypothesis**

$$P(1), P(1) \rightarrow P(2), P(2) \rightarrow P(3), \dots \Rightarrow \forall n \geq 1 P(n)$$

Note: Basis Step can be any integer number m (even negative), just notice that you have to show that

$$\forall n \geq m: P(n) \Rightarrow P(n + 1)$$

Well-Ordering Principle

◆ Any non-empty subset of $\mathbb{N}$ has a minimum element

Example

- the minimum of prime numbers is 2
- The minimum of odd numbers is 1

Induction Principle can be proved by Well-Ordering Principle and vice versa

- Consider $A = \{n: P(n) \text{ is not true}\}$
- Let m be the smallest element of A
- Since 1 is not in A , then $m > 1$ and $P(m-1)$ is true
- Since $P(m-1) \Rightarrow P(m)$, and $P(m-1)$, then $P(m)$ is true
- Therefore, $m \notin A$ contradicting $m \in A$

More Examples

Def: $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$

Problem: $H_{2^n} \geq 1 + \frac{n}{2}$

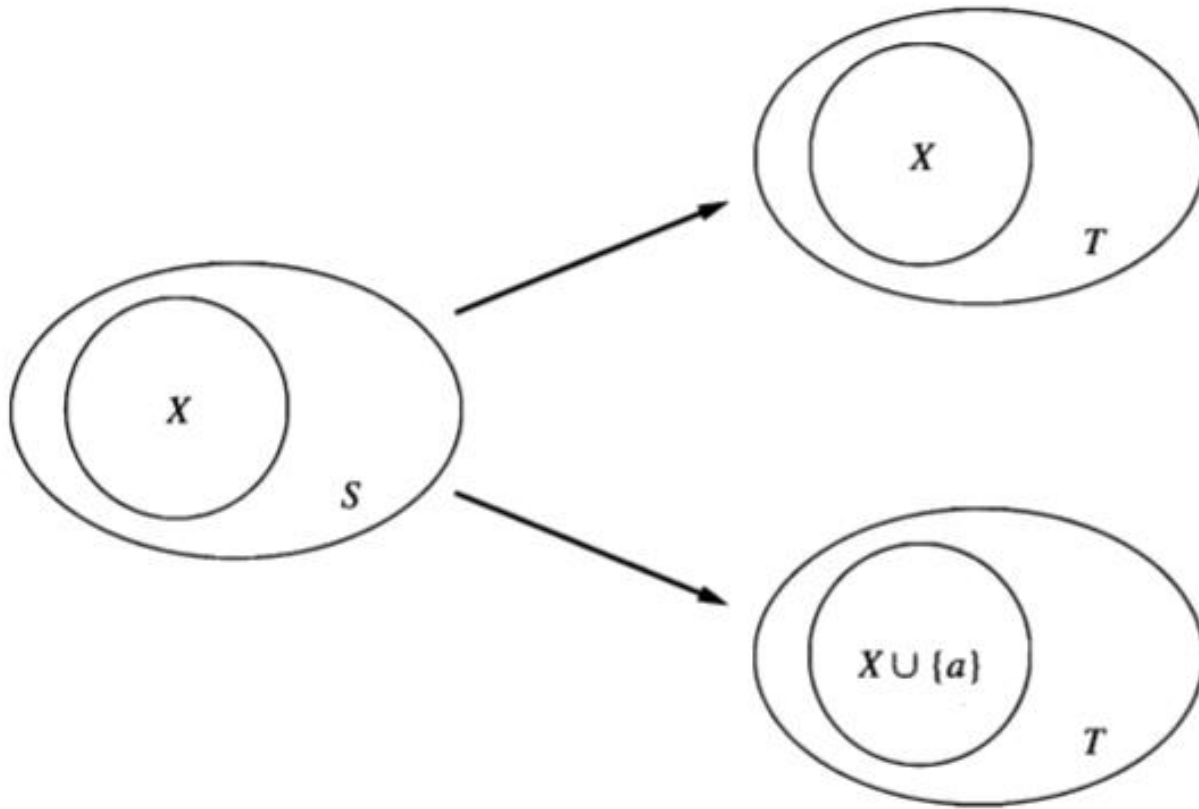
Solution:

- Basis Step: $1 + \frac{1}{2} \geq 1 + \frac{1}{2}$
- Inductive Step: $H_{2^n} \geq 1 + \frac{n}{2} \Rightarrow H_{2^{n+1}} \geq 1 + \frac{n+1}{2}$

$$\begin{aligned} H_{2^{n+1}} &= H_{2^n} + \left(\frac{1}{2^n + 1} + \dots + \frac{1}{2^{n+1}} \right) \geq 1 + \frac{n}{2} + \frac{2^n}{2^{n+1}} \\ &= 1 + \frac{n+1}{2} \end{aligned}$$

More Examples

- Problem: #subsets of $T = 2^{|T|}$
- Solution: $T = S \cup \{a\}$



More Examples

- Problem: De Morgan's laws $\overline{\bigcap A_i} = \bigcup \overline{A_i}$

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \overline{\left(\bigcap_{j=1}^k A_j \right) \cap A_{k+1}}$$

by the definition of intersection

$$= \overline{\left(\bigcap_{j=1}^k A_j \right) \cup \overline{A_{k+1}}}$$

by De Morgan's law (where the two sets are $\bigcap_{j=1}^k A_j$ and A_{k+1})

$$= \left(\bigcup_{j=1}^k \overline{A_j} \right) \cup \overline{A_{k+1}}$$

by the inductive hypothesis

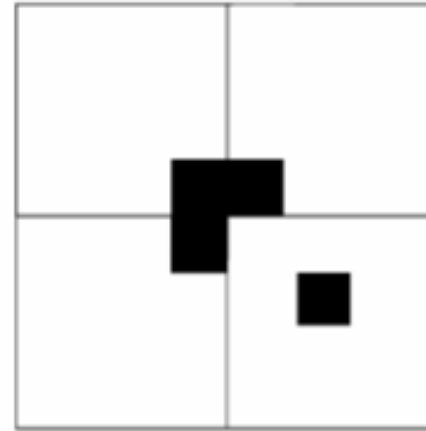
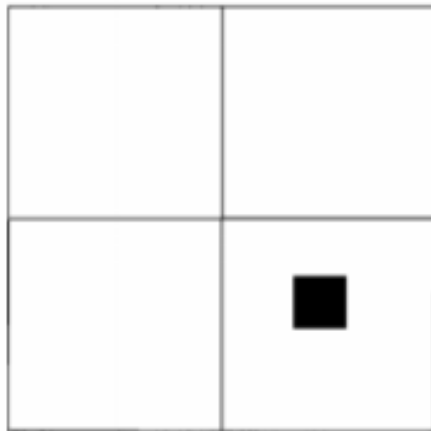
$$= \bigcup_{j=1}^{k+1} \overline{A_j}$$

by the definition of union.

More Examples

Problem: Tiling a $2^n \times 2^n$ board with one arbitrary cell removed with right triominoes

Solution:



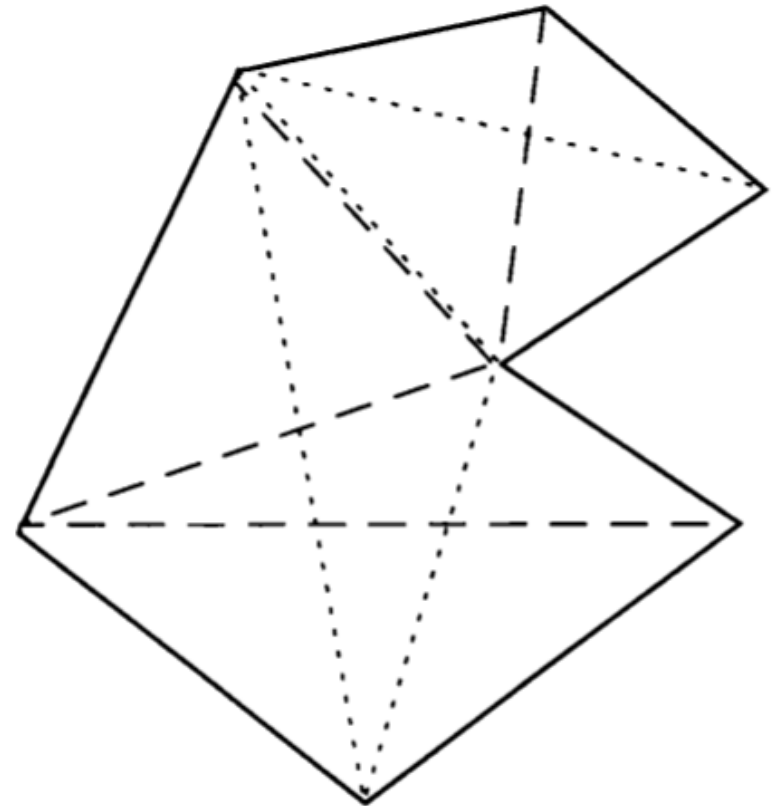
Strong Induction

- Basis Step: $P(1)$ is true
- Inductive Step: $P(1) \wedge P(2) \wedge \dots \wedge P(k) \Rightarrow P(k+1)$

Example

Problem: A simple polygon with n vertices can be partitioned into triangles whose vertices are vertices of the simple polygon.

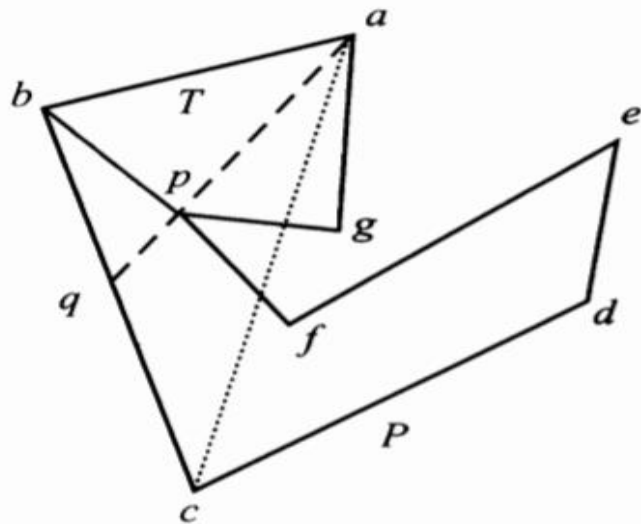
Two Triangulations
Denoted by
Dots and Dashes



Example (Cont)

Solution:

- there is vertices p and q such that the segment pq is inside the polygon.
- Decompose the polygon into two polygon with fewer vertices by drawing pq
- Triangulate each subpolygon



T is the triangle abc

p is the vertex of P inside T such that the $\angle bap$ is smallest

bp must be an interior diagonal of P

Faulty Proof

Problem: $P(n)$: Any set of n non-parallel lines meet in a common point

Solution:

- Basis Step: $P(2)$ is true
- Inductive Step: $P(k) \Rightarrow P(k+1)$

$\{l_1, l_2, \dots, l_{k-1}, l_k\}$ meet at intersection of l_1, l_2

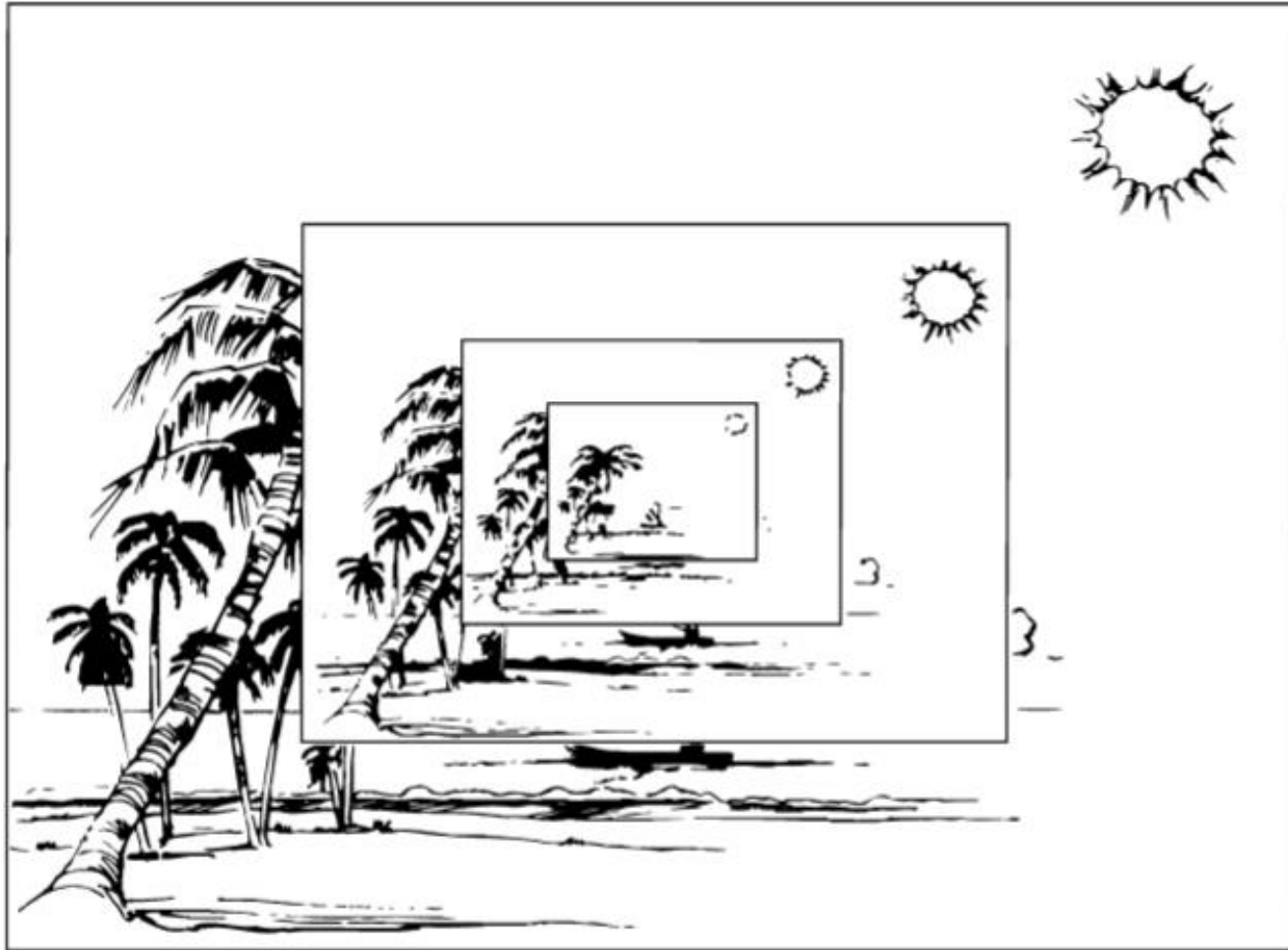
$\{l_1, l_2, \dots, l_{k-1}, l_{k+1}\}$ meet at intersection of l_1, l_2

Then $\{l_1, l_2, \dots, l_{k-1}, l_k, l_{k+1}\}$ meet at intersection of l_1, l_2

Above argument is true when k is at least 3. Then $P(3)$ is basis step which of course is false.

Recursive Def. and Structural Indu.

- Recursive Def: Defining an object in terms of itself



Functions

Fibonacci Numbers

- $f(n+1) = f(n) + f(n-1), f(1) = f(2) = 1$

Problem: $f(n) > \alpha^{n-2}$ where $\alpha = (\sqrt{5} + 1)/2$

Solution:

$$\begin{aligned} f(n+1) &= f(n) + f(n-1) > \alpha^{n-2} + \alpha^{n-3} \\ &= \alpha^{n-3}(\alpha + 1) = \alpha^{n-3}\alpha^2 = \alpha^{n-1} \end{aligned}$$

Sets

- Basis Step: $3 \in S$
- Recursive Step: $x \in S, y \in S \Rightarrow x + y \in S$

Problem: Prove $S = \{3k | k \in N\}$

- $A = \{3k | k \in N\}$
- $3 \in S, 3k \in S \Rightarrow 3(k + 1) \in S$
- Then $A \subseteq S$
- Prove $S \subseteq A$

String

Defining Σ^*

- Basis Step: $\lambda \in \Sigma^*$
- Recursive Step: $x \in \Sigma^*, a \in \Sigma \Rightarrow xa \in \Sigma^*$

Defining the length

- $L(\lambda) = 0$
- $L(xa) = L(x) + 1$

Problem: $L(xy) = L(x) + L(y)$

- Basis: $L(x\lambda) = L(x) + L(\lambda) = L(x)$
- Inductive Step: $y = wa, w \in \Sigma^*, a \in \Sigma$
$$L(xy) = L(xwa) = L(xw) + 1 = L(x) + L(w) + 1$$
$$= L(x) + L(y)$$

Propositional expression

- Basis Step: T, F, and s (a propositional variable)
- Recursive Step:

$$E, F \Rightarrow (\sim E), (E \rightarrow F), (E \wedge F), (E \vee F), (E \leftrightarrow F)$$

Problem: Show any expression contains an equal number of left and right parentheses

Rooted Trees

- Basis Step: A single node is a rooted tree
- Recursive Step: If $T_1, T_2, \dots, T_k$ are rooted trees with roots of $r_1, r_2, \dots, r_k$, then a tree with root r and edges from r to $r_1, r_2, \dots, r_k$ is a rooted tree

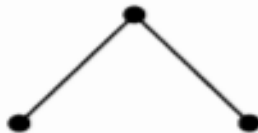
Fully Binary Tree

- Basis Step: A single node is a fully binary tree
- Recursive Step: If T_1, T_2 are fully binary trees with roots of r_1, r_2 , then a tree with root r and edges from r to r_1, r_2 is a fully binary tree

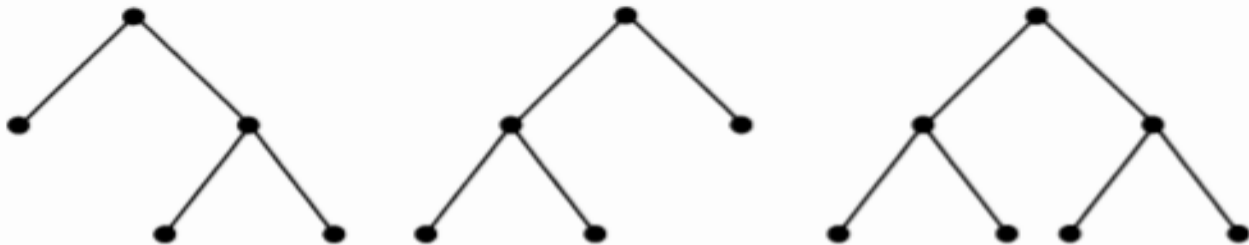
Basis step



Step 1



Step 2



Full Binary Tree

Height ($h(T)$)

- Basis Step: If T is a single node, $h(T)=0$
- Recursive Step: If T constructed from T_1, T_2 , then $h(T)=\max(h(T_1), h(T_2))+1$

Size or the number of nodes ($n(T)$)

Problem: $n(T) \leq 2^{h(T)+1} - 1$

$$\begin{aligned} n(T) &= 1 + n(T_1) + n(T_2) \\ &\leq 1 + (2^{h(T_1)+1} - 1) + (2^{h(T_2)+1} - 1) \\ &\leq 1 + (2^{h(T)} - 1) + (2^{h(T)} - 1) = (2^{h(T)+1} - 1) \end{aligned}$$

Multi-variables

For Fibonacci Num:

$$F(n + m) = F(n + 1)F(m) + F(n)F(m - 1)$$

Fix n and induction on m

- Basis Step:

$$F(n + 1) = F(n + 1)F(1) + F(n)F(0) = F(n + 1)$$

- Inductive Step

$$\begin{aligned} F(n + m) &= F((n + 1) + (m - 1)) \\ &= F(n + 2)F(m - 1) + F(n + 1)F(m - 2) \\ &= (F(n + 1) + F(n))F(m - 1) + F(n + 1)F(m - 2) \\ &= F(n + 1)(F(m - 1) + F(m - 2)) + F(n)F(m - 1) \\ &= F(n + 1)F(m) + F(n)F(m - 1) \end{aligned}$$

Multi-variables

For Fibonacci Num:

$$F(n + m) = F(n + 1)F(m) + F(n)F(m - 1)$$

- induction on $n + m$
- Basis Step:

For $n + m < 2$ check

$$F(n + m) = F(n + 1)F(m) + F(n)F(m - 1)$$

- Inductive Step:

$$\begin{aligned} F(n + m) &= F(n + m - 1) + F(n + m - 2) \\ &= F(n + (m - 1)) + F(n + (m - 2)) \\ &= F(n + 1)F(m - 1) + F(n)F(m - 2) \\ &\quad + F(n + 1)F(m - 2) + F(n)F(m - 3) \\ &= F(n + 1)(F(m - 1) + F(m - 2)) \\ &\quad + F(n)(F(m - 2) + F(m - 3)) \\ &= F(n + 1)F(m) + F(n)F(m - 1) \end{aligned}$$